The Perron Method

In this lecture we show that one can show existence of solutions using maximum principle alone.

1. The Perron method.

Recall in the last lecture we have shown the existence of solutions to the Laplace equation

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega \tag{1}$$

for the special case of $\Omega = B_R$ by writing down the explicit formula for the solution. Unfortunately, for general Ω this is not possible. The Perron method show the existence of the general problem by combining the explicit formula and the maximum principle.

First we recall the following property of subharmonic functions.

- Let $\Omega \subset \mathbb{R}^n$ and $f: \Omega \mapsto [-\infty, \infty)$ be upper semicontinuous in Ω and $f \not\equiv -\infty$. The function f is called subharmonic in Ω if
  - for all $\Omega' \subset\subset \Omega$, the following property holds:
    If u is harmonic in Ω', and $f \leqslant u$ on $\partial\Omega'$, then $f \leqslant u$ in Ω'.
    or equivalently,
  - $v(x) \leqslant \frac{1}{|B_r|} \int_{B_r(x)} u(y)\, \mathrm{d}y$ for any $r > 0$, $B_r(x) \subset \Omega$, or equivalently
  - $v(x) \leqslant \frac{1}{|\partial B_r|} \int_{\partial B_r(x)} u(w)\, \mathrm{d}S$ for any $r > 0$, $B_r(x) \subset \Omega$, or (not really) equivalently
  - if $\triangle v$ exists, $\triangle v \geqslant 0$.

The idea of the Perron method can be best illustrated via the 1D example: solving

$$u'' = 0 \qquad \text{in } (0,1); \quad u(0) = u_0, u(1) = u_1. \tag{2}$$

The solution is a straight line connecting $(0, u_0)$ and $(1, u_1)$.

Now let U_0 be any function connecting these two points. And we modify U_0 as follows. Take any subinterval (a, b), and replace $U_0|_{(a,b)}$ by the solution of

$$u'' = 0 \qquad \text{in } (a,b); \qquad u(a) = U_0(a),\ u(b) = U_0(b). \tag{3}$$

Doing this again and again, we see that the sequence approaches the straight line, which is the solution we need.

It is hard to make this argument rigorous for general U_0. But if we take U_0 to be subharmonic, or roughly speaking $U_0'' \geqslant 0$, then an important observation is that every modification makes the function larger. On the other hand they are all bounded from above by $\max(u_0, u_1)$. Therefore there must be an upper bound and this upper bound is the solution.

This argument can be generalized to higher dimensional problems. We see that we need to guarantee

1. Modifying any subharmonic function by patching it with harmonic functions;
2. The modified function is still subharmonic;
3. Subharmonic functions is upper-bounded by its boundary values.

They are indeed true.

Lemma 1. *We have*

i. If $v \in C^0(\overline{\Omega})$ is subharmonic and $B_R(y) \subset\subset \Omega$, then the harmonic replacement $\bar{v}$ of v, defiend by

$$\bar{v}(x) \equiv \begin{cases} v(x) & x \in \Omega \backslash B_R(y) \\ \dfrac{R^2 - |x-y|^2}{d\, w_d\, R} \displaystyle\int_{\partial B_R(y)} \frac{v(z)}{|z-x|^n}\, \mathrm{d}S_z & x \in B_R(y) \end{cases} \tag{4}$$

is subharmonic in Ω; Furthermore $v \leqslant \bar{v}$ in Ω.

ii. Let v be subharmonic in Ω, if there is $x_0 \in \Omega$ with $v(x_0) = \sup_\Omega v(x)$, then v is constant. In particular, if $v \in C^0(\overline{\Omega})$, then $v(x) \leqslant \max_{\partial\Omega} v(y)$ for all $x \in \Omega$.

iii. If $v_1, \ldots, v_k$ are subharmonic, so is $v \equiv \max\{v_1, \ldots, v_k\}$.

Proof.

i. It is clear that $v \leqslant \bar{v}$ in Ω. Now we argue that $\bar{v}$ is be subharmonic.

   Take arbitrary $\bar{\Omega}' \subset\subset \Omega$, and let u be harmonic in Ω' with $\bar{v} \leqslant u$ on $\partial\Omega'$. Since $v \leqslant \bar{v}$ in Ω, $v \leqslant u$ on $\partial\Omega'$. Since v is subharmonic, $v \leqslant u$ in the whole $\overline{\Omega'}$ (this implies $\bar{v} \leqslant u$ in $\Omega' \backslash B_R(y)$). In particular, $v \leqslant u$ on $\partial(\Omega' \cap B_R(y))$. Since both u and $\bar{v}$ are harmonic in $\Omega' \cap B_R(y)$, we have $\bar{v} \leqslant u$ in there too.

ii. Left as exercise.

iii. Left as exercise. □

Now we turn to the existence of solutions to the problem

$$\triangle u = 0 \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega. \tag{5}$$

Let S_g be the set of all subharmonic functions which are $C^0(\overline{\Omega})$ and are below g on the boundary. We claim that the upper bound is in fact the solution. First we show that the upper bound is harmonic.

Theorem 2. *Let*

$$u(x) \equiv \sup_{v \in S_g} v(x), \tag{6}$$

then u is harmonic in Ω.

Proof. We devide the argument into several steps.

1. u is well-defined.

   To see this, first note that $v \equiv \min_{\partial\Omega} g$ is subharmonic, so S_φ is not empty; Next note that any $v \in S_g$, we have $v(x) \leqslant \sup_{\partial\Omega} g$ for any $x \in \overline{\Omega}$. Therefore the supreme is well-defined at every point.

2. Now consider any $x \in \Omega$. Let v_n be such that $\lim v_n(x) = u(x)$. By replacing v_k by $\max\{v_1, \ldots, v_k, \inf_{\partial\Omega} g\}$, we can assume that $\{v_k\}$ is monotonically increasing and bounded from below. Denote by B_R the ball $B_R(x) \subset\subset \Omega$.

   Now let $\bar{v}_k$ be the harmonic replacements of v_k. It is easy to see that $\bar{v}_k$ is also monotonically increasing. Since $\bar{v}_k \geqslant v_k$, we must have

$$\lim_{k \nearrow \infty} \bar{v}_k(x) = u(x). \tag{7}$$

3. We show that the limit $v = \lim_{k \nearrow \infty} \bar{v}_k$ is harmonic in $B_{R/2}$. To see this, note that $\bar{v}_k - \bar{v}_l$ is harmonic in B_R for any k, l. Thus Harnack inequality implies the existence of C such that

$$|\bar{v}_k(y) - \bar{v}_l(y)| \leqslant C\, |\bar{v}_k(x) - \bar{v}_l(x)| \tag{8}$$

   for any $y \in B_{R/2}$. As a consequence, the convergence $\bar{v}_k \to v$ is uniform in $B_{R/2}$ and therefore v must be harmonic. [1]

4. We need to show that $v = u$ in $B_{R/2}$. Since u is the supreme, $v \leqslant u$. Now if there is $x' \in B_{R/2}$ such that $u(x') > v(x')$, since $u = \sup_{v \in S_g} v$, there is $\tilde{v}$ subharmonic such that $u(x') > \tilde{v}(x') > v(x')$. Set $w_k = \max\{v_k, \tilde{v}\}$. Then the harmonic replacements $\bar{w}_k$ convergens to a harmonic function w in $B_{R/2}$ satisfying

$$v \leqslant w \leqslant u, \tag{9}$$

   which implies

$$v(x_0) = w(x_0). \tag{10}$$

1. To see this, recall that v is harmonic when it has the mean value property, which is kept by uniform convergence.

On the other hand we have $v(x_0) = u(x_0)$ which gives

$$\frac{1}{|B_{R/2}|} \int_{B_{R/2}} w(y)\,\mathrm{d}y \geqslant \frac{1}{|B_{R/2}|} \int_{B_{R/2}} v(y)\,\mathrm{d}y = v(x_0) = w(x_0) = \frac{1}{|B_{R/2}|} \int_{B_{R/2}} w(y)\,\mathrm{d}y. \tag{11}$$

The only possibility is $v \equiv w$ in $B_{R/2}$. But this gives a contradiction as by assumption

$$w(x') \geqslant \tilde{v}(x') > v(x'). \tag{12}$$

Thus ends the proof. $\square$

The next task is to show that u satisfies the boundary condition. That is

$$\lim_{x \to x_0 \in \partial\Omega} u(x) = g(x_0). \tag{13}$$

We use the concept of a barrier.

Recall that a function f is superharmonic if $-f$ is subharmonic. Note that we can replace all "subharmonic" in the above construction by "superharmonic" and obtain $u(x) = \inf v$ where the infimum is over all superharmonic functions whose boundary value $\geqslant g$.

Definition 3.

a) *Let $\xi \in \partial\Omega$. A function $\beta \in C^0(\overline{\Omega})$ is called a barrier at ξ with respect to Ω if*

  i. *$\beta > 0$ in $\overline{\Omega} \backslash \{\xi\}$, $\beta(\xi) = 0$.*

  ii. *β is superharmonic in Ω.*

b) *$\xi \in \partial\Omega$ is called regular if there existes a barrier β at ξ with respect to Ω.*

Remark 4. If β is a "local" barrier at ξ with respect to $U \cap \Omega$, then for any $B_\rho(\xi) \subset\subset U$,

$$\tilde{\beta} \equiv \begin{cases} m \equiv \inf_{U \backslash B_\rho(\xi)} \beta & x \in \bar{\Omega} \backslash B_\rho(\xi) \\ \min(m, \beta) & x \in \bar{\Omega} \cap B_\rho(\xi) \end{cases} \tag{14}$$

is a barrier at ξ with respect to Ω.

To see this, take any $\Omega' \subset\subset \Omega$, and let u be harmonic in Ω' with boundary value $\tilde{\beta}$. We need to show that $u \leqslant \tilde{\beta}$ in Ω'. Since $\tilde{\beta} \leqslant m$, $u \leqslant m$ in Ω'. Next we show $u \leqslant \beta$ in $\Omega' \cap B_\rho(\xi)$. At the boundary, we have

$$u = \tilde{\beta} \leqslant \begin{cases} \min(m, \beta) \leqslant \beta & \text{on } \partial\Omega' \cap B_\rho \\ m = \inf_{U \backslash B_\rho(\xi)} \beta & \text{on } \partial B_\rho \cap \Omega' \end{cases} \Longrightarrow u \leqslant \beta \text{ in } \Omega' \cap B_\rho(\xi). \tag{15}$$

Lemma 5. *If ξ is a regular point of $\partial\Omega$ and g is continuous at ξ, then u takes g as its boundary value at ξ, that is*

$$\lim_{x \to \xi} u(x) = g(\xi). \tag{16}$$

Proof. The idea is the construct super and subharmonic function from g and the barrier β, which gives lower and upper bounds of u.

For any $\varepsilon > 0$, there is $\delta > 0$ such that

$$\begin{array}{ll} |g(x) - g(\xi)| < \varepsilon & \text{for all } |x - \xi| < \delta; \\ c\,\beta(x) \geqslant 2\,M \equiv 2 \sup_{\partial\Omega} |\varphi| & \text{for all } |x - \xi| \geqslant \delta. \end{array} \tag{17}$$

Now consider

$$\bar{v} \equiv g(\xi) + \varepsilon + c\,\beta(x), \qquad \underline{v} \equiv g(\xi) - \varepsilon - c\,\beta(x). \tag{18}$$

We have

$$\underline{v} \leqslant u \leqslant \bar{v} \tag{19}$$

which gives the convergence estimate. $\square$

Summarizing, we obtain

Theorem 6. *Let $\Omega \subset \mathbb{R}^n$ be bounded. The Dirichlet problem*

$$\triangle u = 0 \quad in\ \Omega, \qquad u = g \quad on\ \partial\Omega \tag{20}$$

is solvable for all continuous boundary values g if and only if all points $\xi \in \partial\Omega$ are regular.

Proof. We have already done the "if" part. For the "only if" part, we take g_0 be such that $g(\xi) = 0$ while $g(x) > 0$ for all other $x \in \partial\Omega$. The solution u_0 is a barrier due to strong maximum principle. $\square$

2. The alternating method of H. A. Schwartz.

It is important from the numerical point of view and is the foundation of domain decomposition method.

Theorem 7. *Let Ω_1 and Ω_2 be bounded domains all of whose boundary points are regular for the Dirichlet problem. Suppose that $\Omega_1 \cap \Omega_2 \neq \phi$ and that Ω_1 and Ω_2 are of class C^1 in some neighborhood of $\partial\Omega_1 \cap \partial\Omega_2$, and that they intersect at a nonzero angle. Then the Dirichlet problem fo the Laplace equation on $\Omega \equiv \Omega_1 \cup \Omega_2$ is solvable for any continuous boundary values.*

Proof. We denote $\gamma_1 = \partial\Omega_1 \cap \Omega_2$, $\gamma_2 = \partial\Omega_2 \cap \Omega_1$, $\Gamma_1 = \partial\Omega_1 \backslash \gamma_1$, $\Gamma_2 = \partial\Omega_2 \backslash \gamma_2$.

1. Let $M \equiv \sup_{\partial\Omega} g$ and $m \equiv \inf_{\partial\Omega} g$. We solve the Dirichlet problem[2]

$$\triangle u = 0 \qquad \text{in } \Omega_1, \qquad u = \begin{cases} g & \Gamma_1 \\ M & \gamma_1 \end{cases}. \tag{21}$$

   and obtain u_1, defined on Ω_1.

   Next we solve

$$\triangle u = 0 \qquad \text{in } \Omega_2, \qquad u = \begin{cases} g & \Gamma_2 \\ u_1 & \gamma_2 \end{cases}. \tag{22}$$

   and obtain u_2.

   This process can be continued to obtain u_3, u_4, ... with u_{2k+1} defined on Ω_1 and u_{2k} defined on Ω_2.

2. By strong maximum principle we have $u_2 \leqslant M$ and in particular $u_2 \leqslant u_1$ on γ_2. This implies

$$u_2 \leqslant u_1 \qquad \text{on } \overline{\Omega_1 \cap \Omega_2}. \tag{23}$$

   Next comparing the boundary values of u_1,and u_3 and using maximum principle, we have

$$u_3 \leqslant u_1 \qquad \text{on } \overline{\Omega_1}. \tag{24}$$

   Comparing the values of u_2 and u_3 on $\gamma_1 \cup \gamma_2$ we have

$$u_3 \leqslant u_2 \qquad \text{on } \overline{\Omega_1 \cap \Omega_2}. \tag{25}$$

   Comparing the values of u_2 and u_4 on $\partial\Omega_2 = \Gamma_2 \cup \gamma_2$ we obtain

$$u_4 \leqslant u_2 \qquad \text{on } \overline{\Omega_2}. \tag{26}$$

3. In general, we have

$$u_1 \geqslant u_3 \geqslant \cdots \qquad \text{on } \overline{\Omega_1}; \quad u_2 \geqslant u_4 \geqslant \cdots \quad \text{on } \overline{\Omega_2}; \qquad u_1 \geqslant u_2 \geqslant \cdots \qquad \text{on } \overline{\Omega_1 \cap \Omega_2}. \tag{27}$$

   Therefore there is u on $\Omega_1 \cup \Omega_2$ such that

$$u = \begin{cases} \lim u_{2k+1} & \Omega_1 \\ \lim u_{2k} & \Omega_2 \end{cases}. \tag{28}$$

2. Note that the boundary value here may not be continuous. But it is still bounded and the Perron construction still works with convergence to boundary values only fail at discontinuous points.

Using Harnack inequality we can conclude that the convergence is locally uniform which means u is harmonic.

4. Next we show that u is continuous up to the boundary except for those points in $\partial\Omega_1 \cap \partial\Omega_2$. We can start from

$$\triangle v_1 = 0 \qquad \text{in } \Omega_1, \quad v_1 = \begin{cases} g & \Gamma_1 \\ m & \gamma_1 \end{cases} \tag{29}$$

and obtain a increasing sequence $\{v_n\}$. It is clear that $v_n \leqslant u_n$ and therefore u is between each pair of (v_n, u_n). The continuity of v_n, u_n up to boundary then gives the continuity of u, except for those points in $\partial\Omega_1 \cap \partial\Omega_2$.

5. To show continuity at $\partial\Omega_1 \cap \partial\Omega_2$, one has to modify the iteration process. Instead of using M on γ_1 to obtain u_1, we use a continuous extension of g. This way we can still obtain $\{u_n\}$ but the monotonicity relation does not hold anymore. Instead, we show that

$$\frac{(u_{2n+3} - u_{2n+1})}{\max_{\gamma_1} |u_{2n+2} - u_{2n}|} \leqslant q < 1, \qquad \text{on } \gamma_2 \tag{30}$$

for a uniform q. This, combined with the maximum principle, implies $u_{n+1} - u_n$ converges to 0 as fast as the geometric series q^n, which in turn gives uniform convergence of the sequence.

For details of the last step, see §3.3, 3.4 of J. Jost **Partial Differential Equations**, GTM 214. [3] □

Further readings.

- J. Jost, **Partial Differential Equations**, Chap. 3.

Exercise.

Exercise 1. Let v be subharmonic in Ω, if there is $x_0 \in \Omega$ with $v(x_0) = \sup_\Omega v(x)$, then v is constant (If you are not comfortable with upper semicontinuity, just assume v to be continuous). In particular, if $v \in C^0(\overline{\Omega})$, then $v(x) \leqslant \max_{\partial\Omega} v(y)$ for all $x \in \Omega$.

Exercise 2. Prove: If $v_1, \ldots, v_k$ are subharmonic in Ω, so is $v \equiv \max\{v_1, \ldots, v_k\}$.

3. The key lemma is the following: There exists $0 < q < 1$ depending on Ω_1, Ω_2, such that if $w: \overline{\Omega}_1 \mapsto \mathbb{R}$ is harmonic in Ω_1, and continuous on the closure $\overline{\Omega}_1$, with $w = 0$ on Γ_1, $|w| \leqslant 1$ on γ_1, then

$$|w| \leqslant q \qquad \text{on } \gamma_2. \tag{31}$$

A corresponding result holds if Ω_1 is replaced by Ω_2.